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Duality of Orthogonal Polynomials on a Finite Set

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We prove a certain duality relation for orthogonal polynomials defined on a finite set. The result is used in a direct proof of the equivalence of two different ways (using particles or holes) of computing the correlation functions of a discrete orthogonal polynomial ensemble.

KEY WORDS: Discrete orthogonal polynomials; orthogonal polynomial ensembles.

0. INTRODUCTION

This note is about a certain duality of orthogonal polynomials defined on a finite set. If the weights of two systems of orthogonal polynomials are related in a certain way, then the values of the nth polynomial of the first system at the points of the set equal, up to a simple factor, the corresponding values of the $(M-n)$th polynomial of the second system, where M is the cardinality of the underlying finite set.

We formulate the exact result and prove it in Section 1.

In Section 2 we explain the motivation which led to the result. We compare two different ways to compute probabilistic quantities called *correlation functions* in a certain model. The model is a discrete analog of the *orthogonal polynomial ensembles* which appeared for the first time in the random matrix theory, see, e.g., refs. 1–5. Discrete orthogonal polynomial ensembles were discussed in refs. 6–11. The two different ways correspond to descriptions of an ensemble in terms of *particles* or in terms of *holes*. The relation that we prove shows explicitly how the particle-hole duality transforms the underlying orthogonal polynomials.

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The particle-hole duality in orthogonal polynomial ensembles comes up naturally in tiling models (ref. 11, Section 4) and in representation theoretic models (ref. 8, Sections 5 and 11). Our result applies directly in both cases.

In Section 3 we consider two examples when the orthogonal polynomials are classical (Krawtchouk and Hahn polynomials). In these cases the duality provides relations between similar polynomials with different sets of parameters. The relations are also easily verified using known explicit formulas for the polynomials.

I am very grateful to Grigori Olshanski for numerous discussions. I also want to thank Tom Koornwinder for providing me with his computation regarding the Hahn polynomials, see Section 3, and Vyacheslav Spiridonov for referring me to ref. 12.

1. DUALITY

Theorem 1. Let

$$X=\{x_0, x_1,..., x_M\}\subset \mathbb{R}$$

be a finite set of distinct points on the real line, $u(x)$ and $v(x)$ be two positive functions on X such that

$$u(x_k)\, v(x_k)=\frac{1}{\prod_{i\neq k}(x_k-x_i)^2}, \qquad k=0, 1,..., M, \tag{1}$$

and $P_0, P_1,..., P_M$ and $Q_0, Q_1,..., Q_M$ be the systems of orthogonal polynomials on X with respect to the weights $u(x)$ and $v(x)$, respectively,

$$\deg P_i=\deg Q_i=i,$$

$$\sum_{k=0}^{M} P_i(x_k)\, P_j(x_k)\, u(x_k)=\delta_{ij}\, p_i, \qquad \sum_{k=0}^{M} Q_i(x_k)\, Q_j(x_k)\, v(x_k)=\delta_{ij}\, q_i,$$

$$P_i=a_i x^i+\text{lower terms}, \qquad Q_i=b_i x^i+\text{lower terms}.$$

Assume that the polynomials are normalized so that $b_i=p_{M-i}/a_{M-i}$ for all $i=0, 1,..., M$. Then

$$P_i(x)\sqrt{u(x)}=\epsilon(x)\, Q_{M-i}(x)\sqrt{v(x)}, \qquad x\in X,$$

$$a_i b_{M-i}=p_i=q_{M-i}, \qquad i=0, 1,..., M,$$

where

$$\epsilon(x_k) = \operatorname{sgn} \prod_{l \neq k} (x_k - x_l), \qquad k = 0, 1, \ldots, M.$$

Remark. After this paper was completed, I learned from V. P. Spiridonov that a similar result had been proved earlier in ref. 12. The proof and motivation of ref. 12 are, however, rather different.

Proof. Let us start with one system of polynomials, say, $\{P_i\}$, and define a sequence of functions $\{\tilde{Q}_i\}$ on X by the equalities

$$\tilde{Q}_i(x_k) = \epsilon(x_k)\, P_{M-i}(x_k) \sqrt{\frac{u(x_k)}{v(x_k)}} = \prod_{i \neq k} (x_k - x_i) \cdot P_{M-i}(x_k)\, u(x_k).$$

Then

$$\sum_{k=0}^{M} \tilde{Q}_i(x_k)\, \tilde{Q}_j(x_k)\, v(x_k) = \sum_{k=0}^{M} P_{M-i}(x_k)\, P_{M-j}(x_k)\, u(x_k) = \delta_{ij}\, p_{M-i},$$

so the functions $\{\tilde{Q}_i\}_{i=0}^{M}$ are pairwise orthogonal with respect to the weight $v(x)$, and $q_i = \|\tilde{Q}_i\|_v^2 = p_{M-i}$.

Consider the interpolation polynomial $Q_i(x)$ of degree M such that $Q_i(x) = \tilde{Q}_i(x)$ for all $x \in X$. We have (the hat means that the corresponding factor is omitted)

$$\begin{aligned} Q_i(x) &= \sum_{m=0}^{M} \tilde{Q}_i(x_m) \frac{(x - x_0) \cdots \widehat{(x - x_m)} \cdots (x - x_M)}{(x_m - x_0) \cdots \widehat{(x_m - x_m)} \cdots (x_m - x_M)} \\ &= \sum_{m=0}^{M} P_{M-i}(x_m)\, u(x_m) \cdot (x - x_0) \cdots \widehat{(x - x_m)} \cdots (x - x_M). \end{aligned}$$

The coefficient of x^n of such polynomial equals

$$(-1)^{M-n} \sum_{m=0}^{M} P_{M-i}(x_m)\, u(x_m)\, e_{M-n}(x_0, \ldots, \widehat{x_m}, \ldots, x_M)$$

where e_s are the elementary symmetric functions:

$$e_s(y_0, y_1, \ldots) = \sum_{0 \leqslant i_1 < i_2 < \cdots < i_s} y_{i_1}\, y_{i_2} \cdots y_{i_s}.$$

Denote $e_s(x_0,\dots, x_M)$ by E_s. Note that $E_0 = 1$ by definition. An application of the inclusion–exclusion principle shows that

$$e_s(x_0,\dots, \widehat{x_m},\dots, x_M) = E_s - x_m E_{s-1} + x_m^2 E_{s-2} - \cdots + (-1)^s x_m^s.$$

Then the coefficient of x^n in $Q_i(x)$ equals

$$\begin{aligned}(-1)^{M-n} \sum_{m=0}^{M} & P_{M-i}(x_m)\, u(x_m)(E_{M-n} - x_m E_{M-n-1} + \cdots + (-1)^{M-n} x_m^{M-n}) \\ &= (-1)^{M-n} E_{M-n}\langle P_{M-i}, 1\rangle + (-1)^{M-n-1} E_{M-n-1}\langle P_{M-i}, x\rangle \\ &\quad + \cdots + \langle P_{M-i}, x^{M-n}\rangle.\end{aligned}$$

But the orthogonality of P_j's implies that $\langle P_{M-i}, x^r\rangle = 0$ for $r < M-i$, and

$$\langle P_{M-i}, x^{M-i}\rangle = \frac{\|P_{M-i}\|^2}{a_{M-i}} = \frac{p_{M-i}}{a_{M-i}}.$$

This immediately implies that Q_i is a polynomial of degree i with the leading coefficient $b_i = p_{M-i}/a_{M-i}$. ∎

2. PROBABILISTIC INTERPRETATION

Recall that $X = \{x_0,\dots, x_M\}$ is a finite subset of the real line.

For any $m = 1,\dots, M$, denote by $X^{(m)}$ the set of all subsets of X with m points:

$$X^{(m)} = \{\{x_{i_1},\dots, x_{i_m}\} \mid 0 \leqslant i_1 < \cdots < i_m \leqslant M\}.$$

For any positive function $w(x)$ on X denote by $\mathscr{P}_w^{(m)}$ the probability measure on $X^{(m)}$ defined by the formula:

$$\mathscr{P}_w^{(m)}\{x_{i_1},\dots, x_{i_m}\} = \text{const} \prod_{1\leqslant k<l\leqslant m} (x_{i_k} - x_{i_l})^2 \cdot \prod_{k=1}^{m} w(x_{i_k}).$$

Also denote by $\bar{\mathscr{P}}_w^{(m)}$ the probability measure on $X^{(m)}$ defined by the relation:

$$\bar{\mathscr{P}}_w^{(m)}(A) = \mathscr{P}_w^{(M-m+1)}(X\setminus A), \qquad A \in X^{(m)}.$$

The next claim was essentially proved in ref. 8.

Proposition 2. Let $u(x)$ and $v(x)$ be two positive functions on X satisfying (1). Then $\mathscr{P}_u^{(m)} = \bar{\mathscr{P}}_v^{(M-m+1)}$ for any $m = 1,\dots, M$.

Proof. For arbitrary finite sets B and C we will abbreviate

$$\Pi(B) = \pm \prod_{\substack{x, y \in B \\ x \neq y}} (x-y), \qquad \Pi(B, C) = \prod_{x \in B,\, y \in C} (x-y).$$

The sign of $\Pi(B)$ is inessential.

Take $A = \{x_{i_1}, \ldots, x_{i_m}\} \in X^{(m)}$. We have

$$\mathscr{P}_u^{(m)}(A) = \text{const} \prod_{1 \leqslant k < l \leqslant m} (x_{i_k} - x_{i_l})^2 \cdot \prod_{k=1}^{m} u(x_{i_k}) = \text{const} \cdot \Pi^2(A) \cdot \prod_{x \in A} u(x).$$

Further,

$$\Pi(A) = \pm \Pi(X \setminus A) \cdot \Pi^2(A)\, \Pi(A, X \setminus A) \cdot \frac{1}{\Pi(A)\, \Pi(X \setminus A)\, \Pi(A, X \setminus A)}.$$

But $\Pi(A)\, \Pi(X \setminus A)\, \Pi(A, X \setminus A) = \Pi(X) = \text{const}$, and

$$\Pi^2(A)\, \Pi(A, X \setminus A) = \pm \prod_{x \in A} \left(\prod_{\substack{y \in X \\ y \neq x}} (y - x) \right).$$

Hence, using (1), we get

$$\begin{aligned} \Pi^2(A) \cdot \prod_{x \in A} u(x) &= \text{const} \cdot \Pi^2(X \setminus A) \left(\prod_{x \in A} v(x) \right)^{-1} \\ &= \text{const} \cdot \Pi^2(X \setminus A) \cdot \frac{\prod_{x \in X \setminus A} v(x)}{\prod_{x \in X} v(x)} \\ &= \text{const}' \cdot \Pi^2(X \setminus A) \cdot \prod_{x \in X \setminus A} v(x), \end{aligned}$$

where $\text{const}' = \text{const} \cdot (\prod_{x \in X} v(x))^{-1}$. Thus, $\mathscr{P}_u^{(m)}$ and $\overline{\mathscr{P}}_v^{(M-m+1)}$ differ by a multiplicative constant. Since both $\mathscr{P}_u^{(m)}$ and $\overline{\mathscr{P}}_v^{(M-m+1)}$ are probability measures, they must coincide. ∎

Let μ be an arbitrary probability measure on the set of all subsets of X. Note that any probability measure on $X^{(m)}$ can be trivially extended to a measure on the set of all subsets of X.

For any $n = 1, 2, \ldots, M$, we define the nth correlation function of μ

$$\rho_n(\cdot \mid \mu) \colon X^{(n)} \to \mathbb{R}_{\geqslant 0}$$

by the formula

$$\rho_n(A \mid \mu) = \sum_{B \supset A} \mu(B).$$

In other words, $\rho_n(A \mid \mu)$ is the probability (with respect to μ) that the random set B contains a fixed set $A \in X^{(n)}$.

Below we use the notation of Theorem 1 for the orthogonal polynomials associated with the weights $u(x)$ and $v(x)$.

Proposition 3. For any $m = 1,\dots, M$, the correlation functions of $\mathscr{P}_u^{(m)}$ have the form

$$\rho_n(\{x_{i_1},\dots, x_{i_n}\} \mid \mathscr{P}_u^{(m)}) = \det[K_u^{(m)}(x_{i_k}, x_{i_l})]_{k,l=1,\dots,n},$$

where

$$K_u^{(m)}(x, y) = \sqrt{u(x)\, u(y)} \sum_{i=0}^{m-1} \frac{P_i(x)\, P_i(y)}{p_i}.$$

Proof. A standard argument from the random matrix theory, see, e.g., ref. 1 and ref. 4, Section 5.2. ▮

Note that if $n > m$ then the nth correlation function of $\mathscr{P}_u^{(m)}$ vanishes identically. Indeed, all sets with more than m points have measure zero with respect to $\mathscr{P}_u^{(m)}$. Another way to see the vanishing is to observe that the matrix $\|K_u^{(m)}(x_i, x_j)\|_{i,j=0,\dots,M}$ has rank m. Thus, its $n \times n$ minors expressing $\rho_n(\cdot \mid \mathscr{P}_u^{(m)})$ must vanish if $n > m$.

Similarly, for any $m = 1,\dots, M$, the correlation functions of $\mathscr{P}_v^{(m)}$ have the form

$$\rho_n(\{x_{i_1},\dots, x_{i_n}\} \mid \mathscr{P}_v^{(m)}) = \det[K_v^{(m)}(x_{i_k}, x_{i_l})]_{k,l=1,\dots,n},$$

where

$$K_v^{(m)}(x, y) = \sqrt{v(x)\, v(y)} \sum_{i=0}^{m-1} \frac{Q_i(x)\, Q_i(y)}{q_i}.$$

The determinantal formulas for the correlation functions above imply that $\mathscr{P}_u^{(m)}$ and $\mathscr{P}_v^{(m)}$ belong to the class of *determinantal point processes*, see refs. 13; 14, Section 5.4; 15, Appendix; and 16 for a general discussion of such processes.

Proposition 4. For any $m = 1,\dots, M$, the correlation functions of $\bar{\mathscr{P}}_u^{(m)}$ have the form

$$\rho_n(\{x_{i_1},\dots, x_{i_n}\} \mid \bar{\mathscr{P}}_u^{(m)}) = \det[\bar{K}_u^{(m)}(x_{i_k}, x_{i_l})]_{k,l=1,\dots,n},$$

where

$$\bar{K}_u^{(m)}(x, y) = \delta_{xy} - K_u^{(m)}(x, y).$$

Here δ_{xy} is the Kronecker delta.

Proof. By the definition of $\bar{\mathscr{P}}_u^{(m)}$, we have

$$\rho_n(A \mid \bar{\mathscr{P}}_u^{(m)}) = \sum_{B \supset A} \mathscr{P}_u^{(m)}(X \setminus B) = \sum_{\substack{C \subset X \\ C \cap A = \varnothing}} \mathscr{P}_u^{(m)}(C).$$

The inclusion–exclusion principle, see, e.g., ref. 17, Section 2.1, gives

$$\sum_{\substack{C \subset X \\ C \cap A = \varnothing}} \mathscr{P}_u^{(m)}(C) = \sum_{D \subset A} (-1)^{|D|} \rho_{|D|}(D \mid \mathscr{P}_u^{(m)}).$$

By Proposition 3, the expression on the right-hand side is equal to the alternating sum of all diagonal minors of the matrix $\|K_u^{(m)}(x, y)\|_{x, y \in A}$. By linear algebra, this is equal to $\det[\delta_{xy} - K_u^{(m)}(x, y)]_{x, y \in A}$. ∎

Similarly, for any $m = 1,\dots, M$, the correlation functions of $\bar{\mathscr{P}}_v^{(m)}$ have the form

$$\rho_n(\{x_{i_1},\dots, x_{i_n}\} \mid \bar{\mathscr{P}}_v^{(m)}) = \det[\bar{K}_v^{(m)}(x_{i_k}, x_{i_l})]_{k,l=1,\dots,n},$$

where

$$\bar{K}_v^{(m)}(x, y) = \delta_{xy} - K_v^{(m)}(x, y).$$

Proposition 4 is a special case of the *complementation principle* for the discrete determinantal processes which is due to S. Kerov, see ref. 15, Section A.3.

Observe that Proposition 2 and Propositions 3 and 4 with similar statements regarding $\mathscr{P}_v^{(m)}$ and $\bar{\mathscr{P}}_v^{(m)}$, imply that all the diagonal minors of the matrix

$$K_u^{(m)} = \|K_u^{(m)}(x_i, x_j)\|_{i,j=0,\dots,M}$$

are equal to the corresponding diagonal minors of the matrix

$$I-K_v^{(M-m+1)}=\|\delta_{ij}-K_v^{(M-m+1)}(x_i,x_j)\|_{i,j=0,\dots,M}.$$

In particular, the diagonal entries of these two matrices are equal. Looking at 2×2 diagonal minors, we then conclude that

$$K_u^{(m)}(x,y)=\pm K_v^{(M-m+1)}(x,y)$$

for all $x\neq y$, $x,y\in X$. (Here we used the fact that both matrices are symmetric.)

An obvious guess is that the matrices $K_u^{(m)}$ and $I-K_v^{(M-m+1)}$ are conjugate, and the conjugation matrix is diagonal with diagonal entries equal to ± 1. This guess turns out to be correct.

Set

$$D=\operatorname{diag}(\epsilon(x_0),\epsilon(x_1),\dots,\epsilon(x_M)),$$

where $\epsilon(x)$ was defined in Theorem 1.

Theorem 5. Under the above notation, for any $m=0,1,\dots,M$,

$$K_u^{(m)}=D(I-K_v^{(M-m+1)})\,D,$$

where the functions u and v satisfy (1).

Proof. The equality of the diagonal entries was discussed above: it is exactly the equality of the first correlation functions of the processes $\mathscr{P}_u^{(m)}$ and $\bar{\mathscr{P}}_v^{(M-m+1)}$, see Propositions 2–4. To prove the equality of the off-diagonal entries we employ the well-known Christoffel–Darboux formula, see, e.g., ref. 18, which implies that, for $x\neq y$,

$$K_u^{(m)}(x,y)=\sqrt{u(x)\,u(y)}\,\frac{a_{m-1}}{a_m\,p_{m-1}}\,\frac{P_m(x)\,P_{m-1}(y)-P_{m-1}(x)\,P_m(y)}{x-y},$$

$$K_v^{(M-m+1)}(x,y)=\sqrt{v(x)\,v(y)}\,\frac{b_{M-m}}{b_{M-m+1}q_{M-m}}$$
$$\times\frac{Q_{M-m+1}(x)\,Q_{M-m}(y)-Q_{M-m}(x)\,Q_{M-m+1}(y)}{x-y}.$$

Then Theorem 1 immediately implies that $K_u^{(m)}(x,y)=-\epsilon(x)\,\epsilon(y)\times K_v^{(M-m+1)}(x,y)$, and the proof is complete. ∎

3. EXAMPLES

Our main reference for this section is ref. 19. We use it for the notation and data on the classical orthogonal polynomials considered below.

3.1. Krawtchouk Polynomials

Let $X = \{0, 1,..., N\}$, and

$$u(x) = \binom{N}{x} p^x(1-p)^{N-x} = \frac{N!}{x!\,(N-x)!}\, p^x(1-p)^{N-x}, \qquad x \in X, \quad 0 < p < 1.$$

The polynomials orthogonal with the weight $u(x)$ are called the *Krawtchouk polynomials*, see ref. 19, Section 1.10,

$$P_n(x) = K_n(x; p, N), \qquad n = 0, 1,..., N.$$

The leading coefficient a_n of P_n, the square of the norm p_n of P_n, and the explicit formula for P_n are as follows:

$$a_n = \binom{N}{n}^{-1} \frac{(-1)^n}{n!\, p^n}, \qquad p_n = \binom{N}{n}^{-1} \left(\frac{1-p}{p}\right)^n, \qquad P_n(x) = {}_2F_1\left(\begin{matrix} -n, -x \\ -N \end{matrix} \,\middle|\, \frac{1}{p}\right).$$

Observe that

$$\prod_{\substack{y=0,..., N \\ y \neq x}} (x-y)^2 = x!^2\,(N-x)!^2, \qquad x = 0, 1,..., N.$$

Thus, the dual (according to Theorem 1) weight $v(x)$ has the form

$$v(x) = (u(x)\, x!^2\,(N-x)!^2)^{-1} = \frac{1}{N!^2\,(p(1-p))^N} \binom{N}{x} (1-p)^x\, p^{N-x}.$$

We conclude that $Q_n(x) = \text{const}\, K_n(x; 1-p, N)$. An easy calculation shows that the normalization of Theorem 1 implies that

$$\text{const} = (-1)^N\,(1-p)^N\, N!, \qquad Q_n(x) = (-1)^N\,(1-p)^N\, N!\, K_n(x; 1-p, N).$$

Clearly, $\epsilon(x) = (-1)^{N-x}$, and the claim of Theorem 1 takes the form

$$K_n(x; p, N) = (-1)^x \left(\frac{1-p}{p}\right)^x K_{N-n}(x; 1-p, N), \qquad x = 0,..., M. \tag{2}$$

Of course, this identity can be proved directly using the explicit formula for the Krawtchouk polynomials above. One just needs to use the transformation formula

$$ {}_2F_1\left(\begin{matrix} a, b \\ c \end{matrix}\,\Big|\, z\right) = (1-z)^{-b}\, {}_2F_1\left(\begin{matrix} c-a, b \\ c \end{matrix}\,\Big|\, \frac{z}{z-1}\right). $$

3.2. Hahn Polynomials

The computation below was shown to me by T. Koornwinder. Let X be as above, and

$$ u(x) = \binom{\alpha+x}{x}\binom{\beta+N-x}{N-x}, \qquad \alpha, \beta > -1 \quad \text{or} \quad \alpha, \beta < -N. $$

If $\alpha, \beta > -1$ then $u(x) > 0$, if $\alpha, \beta < -N$ then $(-1)^N u(x) > 0$.

The orthogonal polynomials corresponding to this weight are called the *Hahn polynomials*, see ref. 19, Section 1.5,

$$ P_n(x) = H_n(x; \alpha, \beta, N), \qquad n = 0, 1,..., N. $$

The data are as follows:

$$ a_n = \frac{(n+\alpha+\beta+1)_n}{(\alpha+1)_n\,(-N)_n}, \qquad p_n = \frac{(-1)^n\,(n+\alpha+\beta+1)_{N+1}\,(\beta+1)_n\,n!}{(2n+\alpha+\beta+1)(\alpha+1)_n\,(-N)_n\,N!}, $$

$$ P_n(x) = {}_3F_2\left(\begin{matrix} -n, n+\alpha+\beta+1, -x \\ \alpha+1, -N \end{matrix}\,\Big|\, 1\right). $$

The dual weight has the form

$$ \begin{aligned} v(x) &= (u(x)\,x!^2\,(N-x)!^2)^{-1} \\ &= \frac{(-1)^N}{(\alpha+1)_N\,(\beta+1)_N}\binom{(-\beta-N-1)+x}{x}\binom{(-\alpha-N-1)+N-x}{N-x}. \end{aligned} $$

Thus, $Q_n(x) = \text{const}\,H_n(x; -\beta-N-1, -\alpha-N-1, N)$. Computation of the normalization constant yields

$$ \begin{aligned} &\text{const} = (-1)^N\,(\beta+1)^N, \\ &Q_n(x) = (-1)^N\,(\beta+1)^N\,H_n(x; -\beta-N-1, -\alpha-N-1, N). \end{aligned} $$

The claim of Theorem 1 takes the form

$$H_n(x;\alpha,\beta,N)=\frac{(-\beta-N)_x}{(\alpha+1)_x}\,H_{N-n}(x;-\beta-N-1,-\alpha-N-1,N) \tag{3}$$

for all $x=0,1,\dots,N$.

A direct proof of (3) follows from the transformation formula

$${}_3F_2\left(\begin{matrix}a,b,c\\d,e\end{matrix}\,\middle|\,1\right)=\frac{\Gamma(d)\,\Gamma(d+e-a-b-c)}{\Gamma(d+e-a-b)\,\Gamma(d-c)}\,{}_3F_2\left(\begin{matrix}e-a,e-b,c\\d+e-a-b,e\end{matrix}\,\middle|\,1\right),$$

see ref. 20, Section 7.4.4(1) and ref. 21, Section 3.6.

The limit transition $\alpha=pt$, $\beta=(1-p)\,t$, $t\to\infty$, see ref. 19, Section 2.5.3, brings (3) to (2).

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